JOURNAL OF COMPUTATIONAL PHYSICS 70, 521–525 (1987)

Note

Removal of Spurious Modes Encountered in Solving Stability Problems by Spectral Methods

1. INTRODUCTION

An important application of Chebyshev spectral methods is found in the solution of hydrodynamic stability eigenvalue problems. For example, accurate solutions of the Orr–Sommerfeld equation using expansions in Chebyshev polynomials have been reported in [1–4]. Unfortunately, along with the highly accurate computed values of the true modes, there appear spurious unstable modes with large growth rates whose magnitude increases with an increase of the size of the truncated algebraic system of equations. Another example where spurious unstable modes are computed is an initial value problem of a one-dimensional model of Stokes flow [2]. On the other hand, no spurious roots are computed, using the same procedure in [4], when the conditions for draw resonance of a liquid jet are determined [5]. It is evident that the spurious roots result because of the truncation of the differential equations to a finite-dimensional system of algebraic equations. One way to eliminate the spurious roots in hydrodynamics (if they occur) is to use separate expansions for the vorticity and the stream function [2]. However, the size of the resulting algebraic system essentially doubles. In this paper we develop an alternative technique based on the Galerkin method which results in no increase in the size of the algebraic system.

The occurrence of spurious roots in the spectra of solutions of the Orr–Sommerfeld equation has been reported in [2, 4] even though two different approaches were used to implement the spectral method; the τ-method [1, 2] and a Galerkin-type method [4]. These two approaches differ in the assumed representations and the treatment of the boundary conditions. They agree in the treatment of the conditions imposed by the differential system on the expansion coefficients in that inner products with Chebyshev polynomials are employed. It is very well known, however, that results from direct application of the Galerkin method, where the inner products are taken with base functions satisfying all the homogeneous boundary conditions, are free from spurious modes (see [6] for numerous examples). This fact can be used in conjunction with the author's [4] method to arrive at a scheme which enjoys the infinite-order convergence of the Chebyshev-spectral method [1] and is free from spurious eigenmodes. The procedure is developed in this paper and applied to solving the Orr–Sommerfeld equation corresponding to a pair of base flows; plane Poiseuille and Blasius profiles.

0021-9991/87 $3.00

2. Galerkin–Chebyshev Approximations

2.1. *Formulation*

Linear stability of a parallel flow $U(z)$, $-1 \leqslant z \leqslant 1$, is determined according to the eigenvalues, c (the flow is unstable if there is an eigenvalue with positive imaginary part), of the Orr–Sommerfeld equation

$$\psi^{(4)} - 2\alpha^2\psi^{(2)} + \alpha^4\psi - i\alpha R[(U-c)(\psi^{(2)} - \alpha^2\psi) - U^{(2)}\psi] = 0, \tag{2.1}$$

$$\psi = \psi^{(1)} = 0 \qquad \text{at} \quad z = \pm 1, \tag{2.2}$$

where α is the disturbance wavenumber and R is the Reynolds number. Superscript numbers in parenthesis on functions of z indicate derivatives with respect to z. The procedure begins by assuming a representation for the highest derivative in (2.1) in terms of $T_j(z)$,

$$\psi^{(4)}(z) = \sum_{j=0}^{N} a_j T_j(z). \tag{2.3}$$

The representations of the lower derivatives of $\psi(z)$ are found by successive integration of (2.3). The resulting constants of integration are conveniently chosen so that the boundary conditions (2.2) are satisfied. This procedure is explained in detail [4], thus we have

$$\psi^{(\beta)}(z) = \sum_{j=0}^{N+4-\beta} \sum_{i=0}^{N} g_{ji}^{(\beta)} a_i T_j(z), \qquad \beta = 0, 1, 2, 3. \tag{2.4}$$

The conditions imposed on ψ by (2.1) are now derived. After substitution of (2.3) and (2.4) in (2.1) we can take the Chebyshev inner product with T_i, $i = 0, 1, \ldots, N$. This was the procedure used in [4] and led to an algebraic eigenvalue problem with a pair of spurious roots. A strictly Galerkin procedure may be derived as follows. The representation for $\psi(z)$ in (2.4) can be rewritten as

$$\psi(z) = \sum_{i=0}^{N} a_i \phi_i(z), \tag{2.5}$$

where

$$\phi_i(z) = \sum_{j=0}^{N+4} g_{ji}^{(0)} T_j(z). \tag{2.6}$$

Observe that $\phi_i(z)$ can be regarded as a new linearly independent basis (it is easy to show that the rank of $g_{ji}^{(0)}$ is $(N+1)$) which satisfies all the homogeneous boundary conditions in (2.2). We now use (2.3) and (2.4) in (2.1) as well as the expansion

$$U^{(\beta)}(z) = \sum_{n=0}^{N_b+1-\beta} b_n^{(\beta)} T_n(z), \qquad \beta = 0, 1, 2 \tag{2.7}$$

for the base flow and take the Chebyshev inner product with $\phi_i(z)$, $i=0, 1,..., N$, to derive the algebraic system

$$\sum_{k=0}^{N}\left[\frac{1}{i\alpha R}\left\{c_k g_{ki}^{(0)} - 2\alpha^2 \sum_{j=0}^{N+2} c_j g_{jk}^{(2)} g_{ji}^{(0)} + \alpha^4 \sum_{j=0}^{N+2} c_j g_{jk}^{(0)} g_{ji}^{(0)}\right\}\right.$$
$$+ A_{jlk}\left\{\sum_{l=0}^{N_b}\sum_{n=0}^{N+2}\sum_{j=0}^{N+4} -b_l^{(0)} g_{nk}^{(2)} g_{ji}^{(0)}\right.$$
$$+ \sum_{l=0}^{N_b}\sum_{n=0}^{N+4}\sum_{j=0}^{N+4} \alpha^2 b_l^{(0)} g_{nk}^{(0)} g_{ji}^{(0)}$$
$$\left.\left. + \sum_{l=0}^{N_b+2}\sum_{n=0}^{N+4}\sum_{j=0}^{N+4} b_l^{(2)} g_{nk}^{(0)} g_{ji}^{(0)}\right\}\right] a_k$$
$$= -c \sum_{k=0}^{N}\sum_{j=0}^{N+2} c_j g_{ji}^{(0)} (g_{jk}^{(2)} - \alpha^2 g_{jk}^{(0)})\, a_k, \qquad i=0, 1,..., N, \tag{2.8}$$

where $c_0=2$, $c_i=1$; $i \geqslant 1$ and

$$A_{jlk} = \tfrac{1}{2}[c_j\, \delta_{j,l+k} + \delta_{j,l-k} + \delta_{j,k-l}]. \tag{2.9}$$

Equation (2.8) is the algebraic eigenvalue problem. It is readily reduced to that which we derived in [4] by replacing one $g_{ji}^{(0)}$ by a Kronecker delta in each term in (2.8), thus more computations are needed to form (2.8) than its counterpart in [4]. However, the present procedure requires the solution of the same size linear system. As in [4], solution of (2.8) is obtained using the IMSL routine EIGZC.

TABLE I

First Four Eigenvalues of Plane Poiseuille Flow
$\alpha=1$ and $R=10{,}000$[a]

$N+1$	Method of [4]	Present results
20	$0.0978 + i\,20.4$	$0.239 + i\,0.000814$
	$0.0966 + i\,16.8$	$0.960 - i\,0.0206$
	$0.580 + i\,0.0242$	$0.891 - i\,0.0216$
	$0.827 + i\,0.0137$	$0.963 - i\,0.0219$
26	$0.0754 + i\,56.0$	$0.238 + i\,0.00384$
	$0.0747 + i\,48.3$	$0.970 - i\,0.0314$
	$0.237 + i\,0.00366$	$0.930 - i\,0.0317$
	$0.849 - i\,0.0249$	$0.970 - i\,0.0333$
30	$0.0654 + i\,97.6$	$0.238 + i\,0.00372$
	$0.0649 + i\,85.7$	$0.967 - i\,0.0387$
	$0.237 + i\,0.00372$	$0.964 - i\,0.0389$
	$0.970 - i\,0.0352$	$0.948 - i\,0.0413$

[a] Most unstable eigenvalue: $0.238 + i\,0.00374$[1]

2.2. *Plane Poiseuille Flow*

Here $U(z)=1-z^2$, which sets the values of $b_n^{(\beta)}$, $\beta=0, 1, 2$, in (2.7) and (2.8). In Table I we list the first four eigenvalues corresponding to $\alpha=1$ and $R=10{,}000$ at three different values on N. We also show results obtained by the method of [4] as well as the "exact" value for the single unstable mode [1]. From this table it is clear that simply equating coefficients of T_i leads to a pair of spurious roots (one symmetric and the other antisymmetric about $z=0$) while the present technique is free from these eigensolutions.

2.3. *Blasius Profile*

The expansion coefficients in this case ($b_n^{(\beta)}$) are computed by solving the Blasius differential equation (see [4], we set η_e at 10 and N_b at 30). The first four eigenvalues are shown in Table II. We also show results from [4] as well as the "exact" value of the single unstable mode [3]. Again, while two spurious unstable modes result if we take inner products with T_i, the present technique produces no such eigenvalues.

It is important to note that both approaches, [4] and the present Galerkin method require essentially the same number of expansion coefficients (about 4 less than that needed if one uses the τ-method) to produce accurate solutions of the Orr–Sommerfeld equation for either profile. The reason for the occurrence of the spurious roots is not understood. Perhaps more important, it is not clear at all why there are two (and only two) such eigenmodes regardless of either the truncation N or the base flow.

TABLE II

First Four Eigenvalues of Blasius Profile
$\alpha=0.173$ and $R=580$[a]

$N+1$	Method of [4]	Present results
24	$0.573+i\,19.6$	$0.358+i\,0.00322$
	$0.575+i\,16.7$	$1.000-i\,0.000740$
	$0.407+i\,0.0159$	$1.000-i\,0.00160$
	$1.000-i\,0.000741$	$1.000-i\,0.00284$
30	$0.559+i\,46.7$	$0.364+i\,0.00790$
	$0.560+i\,41.0$	$1.000-i\,0.000740$
	$0.365+i\,0.0123$	$1.000-i\,0.00160$
	$1.000-i\,0.000740$	$1.000-i\,0.00284$
36	$0.549+i\,95.2$	$0.364+i\,0.00810$
	$0.550+i\,85.3$	$1.000-i\,0.000740$
	$0.364+i\,0.0792$	$1.000-i\,0.00160$
	$1.000-i\,0.000740$	$1.000-i\,0.00284$

[a] Most unstable eigenvalue: $0.364+i\,0.00796$[3]

3. Conclusions

We have presented a Galerkin–Chebyshev technique which enjoys infinite-order convergence and overcomes the potentially severe problems associated with spurious unstable eigenvalues (especially with initial value problems). Solutions of the Orr–Sommerfeld equations presented are free from the pair of spurious roots encountered with other methods [1–4].

An important observation is that the Chebyshev methods in the present work and [1–4] require the solution of linear systems of nearly equal order to produce accurate solutions. If one is interested in the instability of a two-dimensional flow to two- or three-dimensional perturbations, there will result a coupled linear system of Orr–Sommerfeld-type equations. Application of the methods of [1–4] would lead to a number of spurious modes much larger than two. One may then have the solve very large linear systems to produce accurate solutions. Thus, the present method, which is free from these modes, may prove crucial to the success of such a study. This important topic, which basically led to the present work, is currently under investigation.

Acknowledgment

Funds for the support of this study have been allocated by the NASA-Ames Research Center, Moffett Field, California, under Interchange NCA2-61.

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Received January 10, 1986; revised June 6, 1986

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